LQG Robust Stability

MEM 355 Performance Enhancement of Dynamical Systems

Harry G. Kwatny
Department of Mechanical Engineering & Mechanics
Drexel University

Outline

- Robust Stability
    - Min-Max Control
    - Classical LQG
    - Stability Margins & LTR

Regulation: Steer the system from any initial state to the origin.

Drexel UNIVERSITY

Min-Max Control

$$\dot{x} = Ax + Bu + Ew$$

$$y = Cx$$

$$J(u,w) = \int_0^\infty \left[y^T(t)y(t) + \rho u^T(t)u(t) - \gamma^2 w^T(t)w(t) \right] dt, \qquad \rho, \gamma > 0$$

objective: $\min_u \max_w J(u,w)$

Theorem: Suppose $w(t)$ has bounded energy, $(A,B),(A,E)$ stabilizable, (A,C) detectable, then

the optimal control is $u^*(t) = Kx(t),\ K = -\frac{1}{\rho}B^T S$

the worst case disturbance is $w^*(t) = \frac{1}{\gamma^2}E^T Sx(t)$

S is the unique, symmetric, nonnegative solution of the Riccati equation:

$$A^T S + SA - S\left(\frac{1}{\rho}BB^T - \frac{1}{\gamma^2}EE^T\right)S = -C^T C$$

Drexel UNIVERSITY

Min-Max Hamiltonian Matrix

$$A^T S + SA - S\left(\frac{1}{\rho}BB^T - \frac{1}{\gamma^2}EE^T\right)S = -C^T C \Rightarrow H = \begin{bmatrix} A & \frac{1}{\gamma^2}EE^T - \frac{1}{\rho}BB^T \\ -C^T C & -A^T \end{bmatrix}$$

stabilizability/detectability $\Rightarrow$

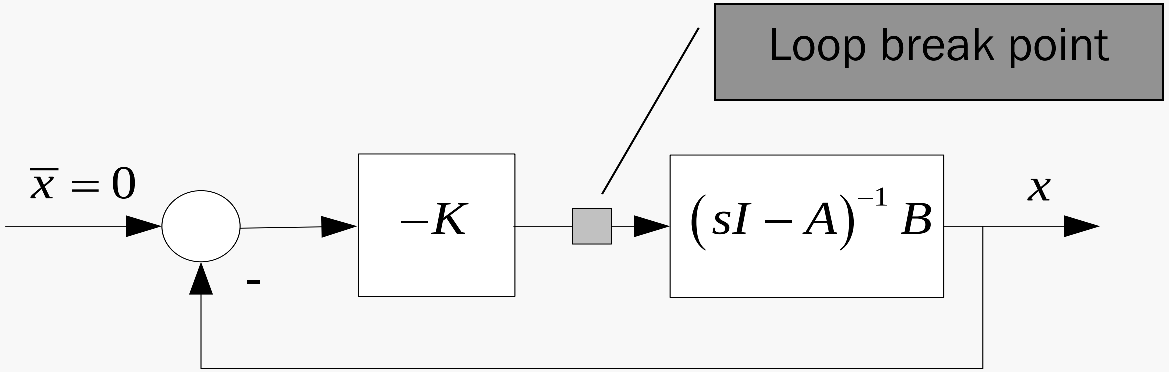
- $\exists \gamma_{\min}$ such that there are no pure imaginary eigenvalues of H if $\gamma > \gamma_{\min}$
- $\gamma \to \infty$ produces LQR solution
- $\gamma = \gamma_{\min}$ is the full state feedback H_∞ solution
- all $\gamma_{\min} \le \gamma < \infty$ are valid min-max controllers

Drexel UNIVERSITY

Min-Max, Continued

- $\operatorname{Re}\lambda(H) \neq 0 \Leftrightarrow \operatorname{Re}\lambda\left(A + \frac{1}{\gamma^2}EE^TS - \frac{1}{\rho}BB^TS\right) < 0$
- The closed loop system matrix is $A - \frac{1}{\rho}BB^TS$, since $\frac{1}{\gamma^2}EE^TS$ is destabilizing, there is some induced stability margin

LQG Robustness: State Feedback



$$G(s) = -K(sI-A)^{-1}B = -K\Phi B, \quad \Phi = (sI-A)^{-1}$$

$$S(s) = \left[I - K(sI-A)^{-1}B\right]^{-1}$$

For R diagonal, $\bar{\sigma}\left(S(j\omega)\right) \le 1\ \forall\omega \Rightarrow$

$$GM: \frac{1}{2}, \infty \quad PM: \frac{\pi}{3} \quad \text{(in each channel)}$$

LQG Robustness: State Feedback 2

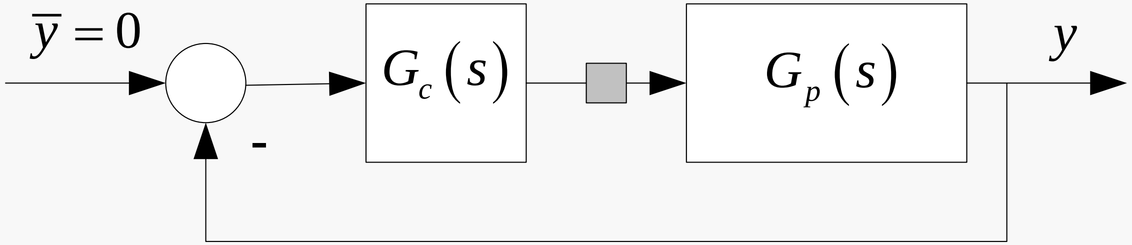
When K is determined via the Ricatti equation, it is known

$$\sigma_{\min}\left[S^{-1}\right]=\min_{\|v\|_R=1}\left\{\left\|\left[I-K\left(j\omega I-A\right)^{-1}B\right]v\right\|_R\right\}\geq 1,\ \omega>0$$

Similarly,

$$\min_{\|v\|_V=1}\left\{\left\|\left[I-C\left(j\omega I-A\right)^{-1}L\right]v\right\|_V\right\}\geq 1,\ \omega>0$$

LQG Robustness: Output Feedback

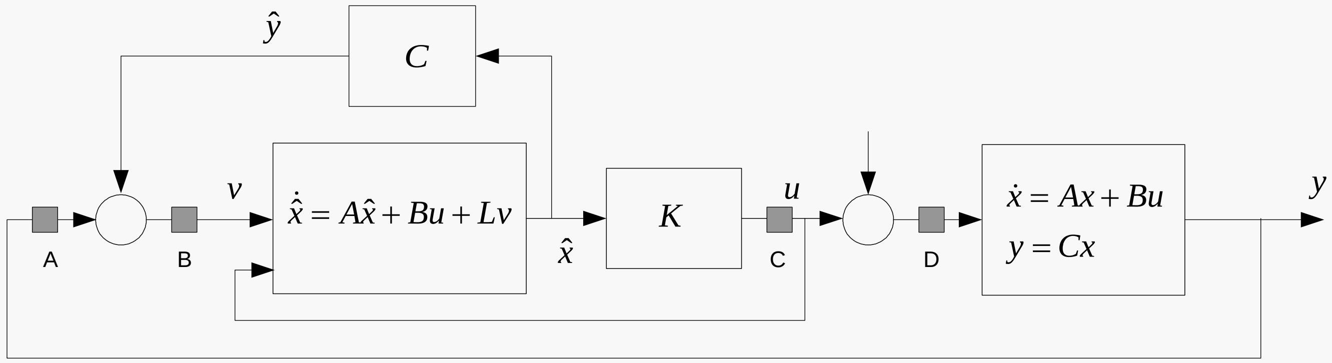


$$G_p(s) = C(sI - A)^{-1} B$$

$$G_c(s) = -K\left[s - (A + BK + LC)\right]^{-1} L$$

No guaranteed margins!

Output Feedback-a closer look



$\Phi(s) \triangleq [sI - A]^{-1}$

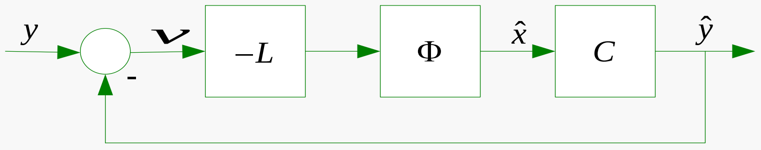
Loop transfer functions:

(A) $C\Phi BK\left[\Phi^{-1} - BK - LC\right]^{-1} L$

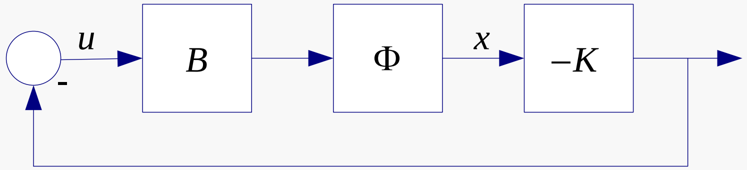
(B) $-C\Phi L$

(C) $-K\Phi B$

(D) $K\left[\Phi^{-1} - BK - LC\right]^{1} LC\Phi B$



"Plant is invisible"



"observer is invisible"

Loop breaking at (A) & (D) do not produce the excellent margins of state feedback.

Example

L. Lublin, S. Grocott, and M. Athans, "H2 and H¥ Control," in *The Control Handbook*, W. S. Levine, Ed. Boca Raton: CRC Press, 1996.

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t)$$

$$y(t) = Cx(t) + 0.1v(t)$$

$$x = \begin{bmatrix} \theta \\ \gamma \\ q \\ V \\ x_w \end{bmatrix} \begin{matrix} \textit{pitch angle} \\ \textit{flight path angle}, \gamma = \theta - \alpha \\ \textit{pitch rate} \\ \textit{speed} \\ \textit{wind gust state} \end{matrix} \qquad u = \begin{bmatrix} \delta_e \\ \delta_f \end{bmatrix} \begin{matrix} \textit{elevator} \\ \textit{flaperon} \end{matrix}$$

$$A = \begin{bmatrix} 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 1.50 & -1.50 & 0.0 & 0.0057 & 1.50 \\ -12.0 & 12.0 & -0.60 & -0.0344 & -12.0 \\ -0.8520 & 0.290 & 0.0 & -0.0140 & -0.290 \\ 0.0 & 0.0 & 0.0 & 0.0 & -0.730 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0 & 0.0 \\ 0.160 & 0.80 \\ -19.0 & -3.0 \\ -0.0115 & -0.0087 \\ 0.0 & 0.0 \end{bmatrix} \quad L = \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 1.1459 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Drexel UNIVERSITY

Loop Transfer Recovery

Recall $L = SC^T V^{-1}$, with

$$SA^T + AS - SC^T V^{-1} CS = -W$$

Let $W = W_0 + \rho^2 BB^T$, ρ a scalar. It can be shown, for a minimum phase plant

$$\lim_{\rho \to \infty} G_c(s) G_p(s) = K(sI - A)^{-1} B \quad \text{complete recovery!}$$

Correspondingly,

$$\lim_{\rho \to \infty} G_c(s) = K(sI - A)^{-1} B G_p^{-1}(s)$$

$\Rightarrow$ some of the estimator poles tend to the plant zeros

For nonminimum phase plants, estimator poles tend to LHP plant zeros, recovery is not complete.